

Non-standard Dirac equations for non-standard spinors

A. G. Nikitin ¹

*Institute of Mathematics, National Academy of Sciences of Ukraine,
3 Tereshchenkivs'ka Street, 01601 Kyiv, Ukraine,*

Abstract

Generalized Dirac equation with operator mass term is presented. Its solutions are non standard (ELKO) spinors which are eigenvectors of the charge conjugation and dual helicity operators. It is demonstrated that in spite of their non covariant nature ELKO can serve as a carrier space of a representation of Poincaré group. However, the corresponding boost generators are not manifestly covariant and generate non-local momentum dependent transformations which are presented explicitly. These results present a new look on group-theoretical grounds of ELKO theories.

¹E-mail: nikitin@imath.kiev.ua

1 Introduction

Dark matter is seemed to be the biggest challenge for human intellect. Being quantitative dominant substance of the universe, it is still waiting for a consistent theoretical framework for its description.

Till now we can indicate only preliminary attempts to create hypothetical elements of future dark matter theory. To create such theory, in essence new classes of fields are requested. One of candidates to form the dark matter is the axion field, see, e.g. [1] and references cited therein. This circumstance was an inspiration for us to analyze group-theoretical grounds of axion electrodynamics and construct exact solutions for the related field equations [2], [3].

Few years ago a new class of spinor fields was introduced in [4], [5]. They are the dual-helicity eigenspinors of the charge conjugation operator (in German: Eigenspinoren des Ladungskonjugationsoperators, ELKO). The concept of ELKO opens new interesting possibilities in constructing of relativistic models, including the models of dark matter (see, e.g., refs [4]-[7]), and of other cosmological phenomena. In particular, this concept was used to provide a new explanation of the accelerated expansion of the universe [8], [9], [10]. Higher dimension aspects of ELKO theory were considered in refs [11] and [12]. We will not discuss the validity and perspectives of all these models, but restrict ourselves to their kinematical grounds connected with using ELKO.

Mathematically, ELKO were put to one of non-equivalent classes of bispinor fields classified by Lounesto [13]. Namely, they were classified as so-called flagpole spinor fields [14]. There exist a clear representation of these spinors proposed in [4] and [5]. Moreover, as it was indicated in [4] and [5], such spinors satisfy the sixteen component Dirac equation supplemented by two additional conditions.

By construction, ELKO are eigenvectors of the dual helicity operator. Such property is not evidently compatible with Lorentz invariance, since this operator is not a relativistic scalar for massive fields. Moreover, in fact ELKO contain a hidden preferred direction that breaks Lorentz symmetry [16].

A natural question arises whether it is possible to formulate the kinematical grounds of ELKO theory in a more compact and relativistic invariant manner. In paper [17] a manifestly covariant generalization of ELKO concept is proposed. The related "dark matter spinors" solve a second order field equations supplemented by nonlocal constraints.

Let us stress that ELKO are only subordinate constructive elements whose connection with physical particle states is realized via quantum field operators. Moreover, the corresponding field equation should be the Klein-Gordon equation, but not the Dirac one, see [4], [5] and paper [15] where the most recent progress on the subject is presented. In other words, the sixteen component Dirac equation for ELKO indicated in [4], [5] is not applied to describe the dynamics of the corresponding fields.

On the other hand, since ELKO are used as expansion coefficients of quantum fields, all properties of these spinors are very interesting and important since they form the grounds of the corresponding field theories. By construction, ELKO satisfies both the Klein-Gordon and (generalized) Dirac equations, and this property should be kept under a transition to a new inertial frame of reference. Thus there are well grounded reasons to study exactly these aspects of ELKO theories, and it was the main motivation for writing the present paper.

We will analyze only kinematical aspects of ELKO without refereing to the corresponding dynamical (field) theories. In other words our research is reduced to studying ELKO spinors.

In the present paper a simple way to describe ELKO by a four-component generalized Dirac equation is presented. More exactly, a modernized Dirac equation will be used, the mass term of which is not proportional to the unit matrix. In addition, a direct and straightforward connection between ELKO and Dirac spinors will be demonstrated.

We also find the explicit form of generators of Poincaré group which can be realized on ELKO. It appears that these generators, like generators of Wigner rotations [18], do not have a manifestly covariant form and generate momentum dependent and so non-local transformations of vectors from their carrier space. These transformations are given explicitly in the present paper. And that is a message that ELKO theory can be treated as Poincaré invariant in spite of that it is not manifestly covariant.

Finally, we present a toy model which, being transparently relativistic invariant, is characterized by the same kinematical equations as ELKO.

2 Multi-component Dirac equation for ELKO

Let us start with the main definitions of ELKO theory. To save a room we will use compact notations presented in what follows.

Like the Dirac spinors, the ELKO $\lambda_{\{+,-\}}^S = \psi_-^+$, $\lambda_{\{-,+\}}^S = \psi_+^+$, $\lambda_{\{+,-\}}^A = \psi_+^-$ and $\lambda_{\{-,+\}}^A = \psi_-^-$ satisfy the Klein-Gordon equation. However, they do not satisfy Dirac equation, which is changed to the following system in the momentum representation [4], [5]:

$$\begin{aligned}\gamma^\mu p_\mu \psi_+^+ + im\psi_-^+ &= 0, \\ \gamma^\mu p_\mu \psi_-^+ - im\psi_+^+ &= 0, \\ \gamma^\mu p_\mu \psi_+^- - im\psi_-^- &= 0, \\ \gamma^\mu p_\mu \psi_-^- + im\psi_+^- &= 0.\end{aligned}\tag{1}$$

Like in [4] and [5] we use the Weyl representation of Dirac matrices with diagonal and hermitian matrix $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Then ELKO can be specified in the following way:

$$\psi_\nu^\varepsilon(\mathbf{p}) = \sqrt{\frac{E+m}{m}} \left(1 - \nu \frac{p}{E+m} \right) \lambda_\nu^\varepsilon\tag{2}$$

where

$$\lambda_\nu^\varepsilon = \begin{pmatrix} \varepsilon \sigma_2 \phi_\nu(0)^* \\ \phi_\nu(0) \end{pmatrix},$$

$\varepsilon, \nu = \pm$, $E = \sqrt{p^2 + m^2}$, $p^2 = p_1^2 + p_2^2 + p_3^2$, σ_2 is the Pauli matrix, and [5]

$$\begin{aligned}\phi_+(0) &= \sqrt{m} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) e^{-i\frac{\varphi}{2}} \\ \sin\left(\frac{\theta}{2}\right) e^{i\frac{\varphi}{2}} \end{pmatrix}, \\ \phi_-(0) &= \sqrt{m} \begin{pmatrix} \sin\left(\frac{\theta}{2}\right) e^{-i\frac{\varphi}{2}} \\ -\cos\left(\frac{\theta}{2}\right) e^{i\frac{\varphi}{2}} \end{pmatrix}\end{aligned}$$

where θ and φ are the polar and azimuthal angles of vector \mathbf{p} .

By construction, the four-component bispinors ψ_ν^ε satisfy the following conditions:

$$C\psi_\nu^\varepsilon = \varepsilon\psi_\nu^\varepsilon\tag{3}$$

and

$$\sigma_p \psi_\nu^\varepsilon = \nu \psi_\nu^\varepsilon \quad (4)$$

where $C = \gamma_2 \kappa$ is the charge conjugation operator, $\kappa \psi = \psi^*$, and $\sigma_p = \frac{1}{p} \gamma_0 \gamma_a p_a \equiv \gamma_5 \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p}$ is a product of the helicity operator with matrix γ_5 . In other words, ψ_ν^ε are eigenvectors of commuting operators C and σ_p , and just equations (1), (3) and (4) can be used as a formal definition of ELKO. Notice that such definition is universal and does not depend on concrete realizations of γ -matrices and spinor components.

Relations (1) specify a sixteen component Dirac equation in momentum representation for $\Psi = \text{column}(\psi_+^+, \psi_-^+, \psi_+^-, \psi_-^-)$, and can be rewritten as:

$$(\Gamma^\mu p_\mu - m)\Psi = 0 \quad (5)$$

where

$$\Gamma^\mu = \begin{pmatrix} 0 & -i\gamma^\mu & 0 & 0 \\ i\gamma^\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & i\gamma^\mu \\ 0 & 0 & -i\gamma^\mu & 0 \end{pmatrix}$$

are the 16×16 Dirac matrices. Moreover, this equation should be considered together with the additional constraints (3) and (4) which reduce the number of independent components of Ψ to 4.

Equations (5) have a symmetric form which is transparently relativistic invariant. The same is true for equation (3). However, it is not the case for the constraint (8) which is not relativistic invariant in the generally accepted meaning. Namely, covariant equations (1) and (5) connect modes ψ_+^ε and ψ_-^ε that are not defined in a covariant manner.

A natural question arises whether ELKO can be treated as a relativistic substance et all. We will see that it is the case since they form a carrier space for a representation of Poincaré group. However, this representation is not manifestly covariant.

One more inspiration to examine the ELKO definition is the general feeling that there are too many equations for a spinor with four independent components. It is naturally to look for a more compact kinematic equation. Just such equation, which also opens a way to give a possible interpretation of Poincaré invariance of ELKO theory, is presented in the following section.

3 Four-component equation for ELKO

Let us consider the following generalized Dirac equation

$$(\gamma^\mu p_\mu + Im)\psi = 0 \quad (6)$$

where I is an involution which commutes with $\gamma^\mu p_\mu$, or pseudo involution anticommuting with $\gamma^\mu p_\mu$. Solutions of any equation of type (6) satisfy the condition

$$(p_0^2 - \mathbf{p}^2 - m^2)\psi = 0 \quad (7)$$

which generate the relativistic dispersion relation. If, in addition, I commutes with generators of Poincaré group, equation (6) is transparently relativistic invariant.

For I being the unity operator equation (6) is reduced to the standard Dirac equation. Choosing $I = \gamma_5$ we obtain a good relativistic equation which is equivalent to the Dirac one.

But there are more involutions I which satisfy the enumerated criteria. In particular, they can be constructed using matrix γ_5 , space inversion P , time reflection T , charge conjugation C and their products.

Let us consider a more sophisticated example of equation (6) with $I = iC\sigma_p$:

$$(\gamma^\mu p_\mu + iC\sigma_p m) \psi = 0. \quad (8)$$

Just this equation can be used to describe the kinematics of ELKO. Indeed, this compact expression is completely equivalent to the cumbersome system (1), (3), (4). To prove this statement let us introduce the following operator

$$P_\nu^\varepsilon = \frac{1}{4}(1 + \varepsilon C)(1 + \nu \sigma_p) \quad (9)$$

where ε and ν are parameters which independently take the values ± 1 . Since C commutes with σ_p , and $C^2 = \sigma_p^2 = 1$ operators (9) satisfy the relations $P_\nu^\varepsilon P_{\nu'}^{\varepsilon'} = \delta^{\varepsilon\varepsilon'} \delta_{\nu\nu'} P_\nu^\varepsilon$ and so are projectors.

Acting on (8) from the left by P_ν^ε and using the following identities:

$$P_\nu^\varepsilon \gamma^\mu p_\mu = \gamma^\mu p_\mu P_{-\nu}^{-\varepsilon}, \quad P_\nu^\varepsilon iC\sigma_p = i\varepsilon \lambda P_\nu^{-\varepsilon},$$

and notations

$$P_\nu^\varepsilon \psi = \psi_\nu^\varepsilon. \quad (10)$$

we immediately come to equations (1), (3) and (4). On the other hand, summing up all equations included into system (1) and using definitions (10) we come to equation (8). Thus the system (1), (3), (4) admits rather compact formulation (8).

Let us show that equation (8) is mathematically equivalent to the Dirac equation. Indeed, multiplying (8) by γ_0 we transform it to the Schrödinger form:

$$p_0 \psi = H \psi, \quad H = \gamma_0 \gamma_a p_a + i\gamma_0 C \sigma_p m. \quad (11)$$

Then, making the transformation

$$\psi \rightarrow \psi_D = U \psi, \quad H \rightarrow H_D = U H U^{-1} \quad (12)$$

with

$$\begin{aligned} U &= \frac{1}{2}(1 - \gamma_5 C \sigma_p)(1 - i\gamma_5), \\ U^{-1} &= \frac{1}{2}(1 + i\gamma_5)(1 + \gamma_5 C \sigma_p) \end{aligned} \quad (13)$$

we reduce (11) to the standard Dirac equation:

$$p_0 \psi_D = H_D \psi_D, \quad H_D = \gamma_0 \gamma_a p_a + \gamma_0 m. \quad (14)$$

Using transformation (12) we can specify ELKO in Dirac representation. Namely, this transformation reduces operators C and σ_p to the following form:

$$C \rightarrow UCU^{-1} = -\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p}, \quad \sigma_p \rightarrow U\sigma_p U^{-1} = \gamma_5 \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p}$$

and so the counterparts of ELKO satisfying (3) and (4) are the Dirac spinors $(\psi_D)_\nu^\varepsilon = U\psi_\nu^\varepsilon$ which are eigenvectors of chirality operator $\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p}$ and matrix γ_5 , which satisfy the following relations:

$$\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p}(\psi_D)_\nu^\varepsilon = -\varepsilon(\psi_D)_\nu^\varepsilon, \quad \gamma_5(\psi_D)_\nu^\varepsilon = -\nu\varepsilon(\psi_D)_\nu^\varepsilon.$$

Operator (13) satisfies the condition $UU^\dagger = 1$ and includes the complex conjugation operation. Following Wigner [19] we classify (12) as an unitary-antiunitary transformation.

4 Relativistic invariance

Equation (8) is completely equivalent to system (1), (3), (4) and generates a relativistic dispersion relation (7). However, the multiplier for the mass term in (8) is not a relativistic scalar, and this fact can be treated as a direct proof that ELKO are not well defined covariant spinors. On the other hand, equation (8) is equivalent to the relativistic Dirac equation, and so it has to inherit its symmetries at least in some more generalized meaning.

In the wide sense, relativistic invariance of a differential equation means that its solutions form a carries space of a representation of Poincaré group. Let us show that equation (8) satisfies this weak invariance condition.

It is a common knowledge that the Dirac equation is relativistic invariant. Moreover, the corresponding generators of Poincaré group can be represented in the following form:

$$\begin{aligned} P_0 &= H_D, \quad P_a = p_a, \\ J_{ab} &= p_b \frac{\partial}{\partial p_a} - p_a \frac{\partial}{\partial p_b} + S_{ab}, \\ J_{0a} &= -p_0 \frac{\partial}{\partial p_a} + S_{0a} \end{aligned} \tag{15}$$

where $p_0 = \pm\sqrt{p^2 + m^2}$ and $S_{\mu\nu} = \frac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$.

On the set of solutions of equation (14) the boost generators J_{0a} can be rewritten in the following form:

$$J_{0a} = -\frac{1}{2}(H_D \hat{x}_a + \hat{x}_a H_D) \tag{16}$$

where $\hat{x}_a = \frac{\partial}{\partial p_a} - \frac{p_a}{p^2}$.

To find a realization of these generators on the set of solutions of equation (8) it is sufficient to make the transformation $P_\mu \rightarrow P'_\mu = U^{-1}P_\mu U$, $J_{\mu\nu} \rightarrow J'_{\mu\nu} = U^{-1}J_{\mu\nu}U$ where P_μ and $J_{\mu\nu}$ with $\mu, \nu = 0, 1, 2, 3$ are generators (15), and U is operator (13). As a result we obtain:

$$P_0 = H, \quad P_a = p_a, \quad J'_{ab} = J_{ab}, \tag{17}$$

$$J'_{0a} = -\frac{1}{2}(\hat{x}_a H + H \hat{x}_a) + \frac{m}{p^2} \varepsilon_{abc} \gamma_b p_c \quad (18)$$

where H is the hamiltonian fixed in (11).

Alternatively, starting with realization (15) for J_{0a} and using the identities

$$\begin{aligned} K_a &= -U^{-1} p_0 \frac{\partial}{\partial p_a} U = -p_0 \left(\frac{\partial}{\partial p_a} + \frac{i}{p^2} S_{ab} p_b (\gamma_5 \sigma_p C - 1) \right), \\ \hat{S}_{0a} &= U^{-1} S_{0a} U = S_{0a} - \frac{i}{p} S_{ab} p_b (\sigma_p - \gamma_5 C) \end{aligned} \quad (19)$$

we obtain:

$$J'_{0a} = K_a + \hat{S}_{0a} = -p_0 \frac{\partial}{\partial p_a} + \Sigma_{0a} \quad (20)$$

where

$$\Sigma_{0a} = S_{0a} + \frac{i}{p^2} S_{ab} p_b (p_0 - \gamma_0 \gamma_a p_a) (1 - \gamma_5 \sigma_p C). \quad (21)$$

Operators (17), (18) and (17), (20) satisfy the following commutation relations

$$\begin{aligned} [P'_\mu, P'_\nu] &= 0, \quad [P'_\mu, J'_{\lambda\sigma}] = g_{\mu\lambda} P'_\sigma - g_{\mu\sigma} P'_\lambda, \\ [J'_{\mu\nu}, J'_{\lambda\sigma}] &= g_{\mu\sigma} J'_{\nu\lambda} + g_{\nu\lambda} J'_{\mu\sigma} - g_{\mu\lambda} J'_{\nu\sigma} - g_{\nu\sigma} J'_{\mu\lambda} \end{aligned}$$

which specify the Lie algebra of Poincaré group.

Thus we find the explicit form of generators of Poincaré group which can be defined on the set of solutions of equation (11) for ELKO. As was expected, the angular momentum operators for ELKO and Dirac fields have the same form.

The boost generators J_{0a} and J'_{0a} are different. Moreover, they are qualitatively different. Indeed, using equation (14) generators J_{0a} can be rewritten in covariant form (15), whereas generators J'_{0a} do not keep this property.

5 Lorentz transformations for ELKO

An important quality of realization (15) is that the matrix term S_{0a} which generates transformations for the wave function commutes with the term $p_0 \frac{\partial}{\partial p_a}$ responsible for transformations of independent variables. As a result Lorentz transformations for Dirac spinors have the following generic form:

$$\psi(\tilde{p}) \rightarrow D(\Lambda^{-1}) \psi(\Lambda \tilde{p}) \quad (22)$$

where $\tilde{p} = (p_0, p_1, p_2, p_3)$, Λ is the Lorentz transformation matrix and $D(\Lambda^{-1})$ is a numeric matrix dependent on transformation parameters. In particular, for Lorentz boost we have

$$D(\Lambda^{-1}) = \exp(S_{0a} \theta_a) = \cosh\left(\frac{\theta}{2}\right) + \frac{2S_{0a} \theta_a}{\theta} \sinh\left(\frac{\theta}{2}\right) \quad (23)$$

where θ_a with $a = 1, 2, 3$ are transformation parameters and $\theta = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}$.

Notice that transformations for the wave function are the same for all values of independent variables.

The boost generator (20) does not have a covariant form, and the matrix term Σ_{0a} is much more complicated than term S_{0a} present in (15). It depends on \mathbf{p} , does not commute with $p_0 \frac{\partial}{\partial p_a}$ and generates dependent on \mathbf{p} transformations for ψ . Nevertheless, integrating the Lie equations generated by operators (20), it is possible to find Lorentz boost transformations for ELKO.

Let us note that in spite of its non-covariant form, boost generator (20) gives rise to covariant transformations (22), (23) provided the new inertial reference frame moves parallel to momentum \mathbf{p} . Indeed, in this case the transformation parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ can be represented as $\boldsymbol{\theta} = \alpha \mathbf{n}$ where $\mathbf{n} = \frac{\mathbf{p}}{p}$, and so $\Sigma_a \theta_a \equiv S_{0a} \theta_a$, exactly as in the case of Dirac equation. Thus it is possible to make standard transformations to the rest frame, study VSR aspects of ELKO [21], etc, etc.

But it is interesting to find generic Lorentz boost for ELKO when transformation parameters are not dependent on momenta. First we note that using (20), it is possible to write the infinitesimal Lorentz boost in the following form:

$$\begin{aligned} p_a &\rightarrow p'_a = (1 - E \frac{\partial}{\partial p_b} \theta_b) p_a = p_a - \theta_a p_0, \\ p_0 &\rightarrow p'_0 = (1 - E \frac{\partial}{\partial p_b} \theta_b) E = p_0 - \theta_a p_a, \\ \psi &\rightarrow \psi' = (1 + \Sigma_{0a} \theta_a) \psi \end{aligned}$$

where θ_a are transformation parameters and Σ_{0a} are matrices (21).

In particular, the spinors ψ_ν^ε defined by relation (10) are transformed as:

$$\psi_\nu^\varepsilon \rightarrow \left(1 - \frac{\varepsilon p_a \theta_a}{2p} \right) \psi_\nu^\varepsilon + \frac{i p_0 S_{ab} p_b \theta_a}{p} \left(\frac{p_0}{p} \psi_\nu^\varepsilon + \left(\frac{p_0}{p} + \nu \right) (\delta_{\varepsilon\nu} \psi_\alpha^\alpha - \psi_\varepsilon^\nu) \right). \quad (24)$$

Starting with (20) it is possible to find also finite Lorentz transformations. To do it it is sufficient to solve the Lie equations for transformations generated by these infinitesimal operators. It is sufficient to restrict ourselves to the particular case $\theta_1 = \theta_2 = 0$, $\theta_3 = \theta$ then the generic Lorentz boost can be obtained by a rotation transformation. In this particular case the Lorentz boost generated by infinitesimal operator (20) has the following form:

$$\begin{aligned} p'_1 &= p_1, \quad p'_2 = p_2, \\ p'_3 &= p_3 \cosh \theta - p_0 \sinh \theta, \\ p'_0 &= p_0 \cosh \theta - p_3 \sinh \theta, \end{aligned} \quad (25)$$

$$\begin{aligned} \psi_\nu'^\varepsilon &= \left(F(\theta)(A_+ + iBS_{3a}p_a) + \sinh \frac{\theta}{2} \frac{\nu}{p} (iA_- S_{3a}p_a - \tilde{p}^2 B) \right) \psi_\nu^\varepsilon \\ &+ \left(F(\theta)(A_- + iBS_{3a}p_a) + \sinh \frac{\theta}{2} \frac{\nu}{p} (iA_+ S_{3a}p_a - \tilde{p}^2 B) \right) \gamma_5 (\delta_{\varepsilon\nu} \psi_\alpha^\alpha - \psi_\varepsilon^\nu) \end{aligned} \quad (26)$$

where

$$F(\theta) = \cosh \frac{\theta}{2} - \sinh \frac{\theta}{2} \frac{\varepsilon p_3}{p}, \quad A_\pm = \frac{\tilde{p}^2 + p_3 p'_3 \pm p p'}{2 p p'}, \quad B = \frac{p'_3 - p_3}{2 p p'},$$

$$\tilde{p}^2 = p_1^2 + p_2^2, \quad p = \sqrt{p_1^2 + p_2^2 + p_3^2}, \quad p' = \sqrt{p_1'^2 + p_2'^2 + p_3'^2}.$$

We will prove rather complicated formulae (26) and (24) in Appendix.

6 One more non-standard Dirac equation

Let us return to equations (6). We will consider their solutions as functions of four independent variables p_0, p_1, p_2 and p_3 which are equal in rights. Then we postulate invariance of (6) with respect to the following discrete transformations

$$\begin{aligned} \Psi(p_0, \mathbf{p}) &\rightarrow P\Psi(p_0, \mathbf{p}) = \gamma_0\Psi(p_0, -\mathbf{p}), \\ \Psi(p_0, \mathbf{p}) &\rightarrow T\Psi(p_0, \mathbf{p}) = \gamma_1\gamma_3\Psi^*(-p_0, \mathbf{p}), \\ \Psi(p_0, \mathbf{p}) &\rightarrow C\Psi(p_0, \mathbf{p}) = \gamma_2\Psi^*(p_0, \mathbf{p}). \end{aligned}$$

By definition, P commutes with $\gamma^\mu p_\mu$ while C and T anticommute with this term. Thus, in order to equation (6) be invariant with respect to these transformations, it is necessary to ask for the following conditions for I :

$$PI = IP, \quad CI = -IC, \quad TI = -IT.$$

In addition, to guarantee correct dispersion relations (7), (pseudo)involution I should satisfy one of the following relation:

$$\gamma^\mu p_\mu I = I\gamma^\mu p_\mu, \quad I^2 = 1 \tag{27}$$

or, alternatively,

$$\gamma^\mu p_\mu I = -I\gamma^\mu p_\mu, \quad I^2 = -1. \tag{28}$$

These conditions together with the requirement of Lorentz invariance leave the only possibility for I , i.e., $I = i\gamma_5 PT$. In this case equation (6) takes the following form:

$$(\gamma^\mu p_\mu + im\gamma_5 PT) \Psi = 0 \tag{29}$$

where we change $\psi \rightarrow \Psi$ to discriminate solutions of (29) from wave functions discussed in the previous sections.

Operator $i\gamma_5 PT$ commutes with $\gamma^\mu p_\mu$ and is an involution, i.e., $(i\gamma_5 PT)^2 = 1$. Thus acting on equation (29) from the left by $(\gamma^\mu p_\mu - im\gamma_5 PT)$ we immediately find that equation (29) generates condition (7). Moreover, in contrast with (11), equation (29) is transparently relativistic invariant.

Let show that there exist some intriguing similarities between solutions of equation (29) and ELKO. Indeed, acting to this equation from the left by the projector

$$\hat{P}_\lambda^\varepsilon = \frac{1}{4}(1 + \varepsilon C)(1 + \lambda\gamma_5 PCT)$$

and using the identities

$$\hat{P}_\lambda^\varepsilon \gamma^\mu p_\mu = \gamma^\mu p_\mu \hat{P}_{-\lambda}^{-\varepsilon}, \quad \hat{P}_\lambda^\varepsilon i\gamma_5 PT = i\varepsilon \lambda \hat{P}_\lambda^{-\varepsilon},$$

one can make sure that the linearly independent functions

$$\Psi_+^+ = \hat{P}_+^+ \Psi, \quad \Psi_-^+ = \hat{P}_-^+ \Psi, \quad \Psi_-^- = \hat{P}_-^- \Psi, \quad \Psi_+^- = \hat{P}_+^- \Psi \quad (30)$$

satisfy the fundamental equations of ELKO theory, given by formulae (1).

In accordance with (30), spinors Ψ_λ^ε satisfy the following conditions:

$$C\Psi_\lambda^\varepsilon = \varepsilon\Psi_\lambda^\varepsilon, \quad \gamma_5 PCT\Psi_\lambda^\varepsilon = \lambda\Psi_\lambda^\varepsilon. \quad (31)$$

Since operator $\gamma_5 PCT$ is nothing but a total reflection of all independent variables, the latter equation can be rewritten in the following form:

$$\Psi_\lambda^\varepsilon(-p_0, -\mathbf{p}) = \lambda\Psi_\lambda^\varepsilon(p_0, \mathbf{p}). \quad (32)$$

Thus, like ELKO, functions (30) are eigenvectors of the charge conjugation operator, satisfying equations (1) and (3). However, in contrast with (4), they are not eigenvectors of the chirality operator, but are eigenvectors of $\gamma_5 PCT$ instead.

The fundamental distinction of the introduced spinors Ψ from ELKO is that, in contrast with (4), both conditions (31) are transparently relativistic invariant.

Finally, let us represent a non-standard Dirac equation in configuration space:

$$(i\gamma^\mu \partial_\mu - imR)\Psi(x) = 0 \quad (33)$$

where $x = (x_0, x_1, x_2, x_3)$ and R is the total reflection operator whose action on $\Psi(x)$ is defined as $R\Psi(x) = \Psi(-x)$.

Acting on (33) from the left by projectors $P_+ = \frac{1}{2}(1 + R)$ and $P_- = \frac{1}{2}(1 - R)$, we obtain the following system

$$\begin{aligned} i\gamma^\mu \partial_\mu \Psi_+(x) &= im\Psi_-(x), \\ i\gamma^\mu \partial_\mu \Psi_-(x) &= -im\Psi_+(x) \end{aligned} \quad (34)$$

where $\Psi_\pm = P_\pm \Psi$ are eigenvectors of the total reflection operator.

Up to the meaning of vectors Ψ_\pm the system (34) coincides with equations for ELKO in configuration space, presented, e.g., in [5].

7 Discussion

In this paper a new look on the kinematical grounds of the ELKO theories is presented. Namely, we give a compact four component formulation (8) of kinematic equation for these spinors. Then, a simple and straightforward connection between Dirac spinors and ELKO is presented. Finally, the transformation properties of ELKO w.r.t. Lorentz boost are discussed. Since ELKO also satisfy equations (1), (3) and (4) by construction, the results of the present paper could be interesting for experts in ELKO approach.

The transformations connecting the Dirac spinors and ELKO were studied in [20]. However, these transformations were made under the supposition that the left handed components of the Dirac and ELKO coincide. In order this supposition to be correct, the Dirac spinors should satisfy one of the additional constraints discussed in [20]. The transformation for ELKO generated by operator (13) is valid without additional constraints.

We show that in spite of that the eigenvectors of dual helicity operator are not covariant subjects, ELKO form a carrier space of the representation of Poincaré group, whose generators are given by equations (17). The corresponding boost generators do not have a covariant form. Nevertheless, they generate covariant transformations for the case when the new frame of reference moves parallel to particle momentum. These facts can be used for justification of ELKO approach which appears to be non-covariant in the standard meaning [16].

Equations, presented in Section 6 are just toy models which are seemed to be rather peculiar. In particular, the equality in rights of all variables in equation (29) is a natural but non-standard proposition. Usually p_0 is considered as a distinguished variable which is not affected by the time reflection.

The formal analogy of these equations with kinematic equations for ELKO is rather curious. And this analogy generates a challenge to search for possible applications of the corresponding fields in non-standard physical theories.

A specific feature of equations (8), (29) and (33) is that they include involutions $C\sigma_p$, $\gamma_5 PCT$ or R as essential constructive elements. Such (and other) involutions present additional tools for creating alternatives to Dirac's factorization of the Klein-Gordon equation. Apparently the first example of such non-standard factorization was proposed long time ago in paper [22] where a two component version of first order equations for a massive spinor field was discussed.

It is interesting to note that equation (11) for ELKO can be decoupled to two subsystems each of which, like equation proposed in [22], is two-component like equation proposed in [22]. Indeed, hamiltonian H commutes with diagonal matrix γ_5 , and so

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

where

$$H_{\pm} = \sigma_a p_a \left(\pm 1 - \frac{im}{p} \sigma_2 \kappa \right)$$

and σ_a are Pauli matrices. Let us note that boost transformations mix eigenvectors of H_+ and H_- .

Involutive discrete symmetries have useful applications in construction of exact Foldy-Wouthuysen transformations [23] and generating of non-standard realizations of symmetry algebras and superalgebras [24], [25], [26], [27], [28]. We see that such involutions can also be effectively used to formulate a compact equation for ELKO.

8 Appendix. Integration of Lie equations

The boost generator (20) includes involutions C and σ_p , and the corresponding Lie equations are rather complicated. We will find the action of this generator on eigenvectors ψ_{ν}^{ε} of these involutions whose formal definition is given by equation (10). Let us define the corresponding matrix entries of boost operator (20):

$$(J'_{0a})^{\varepsilon\varepsilon'}_{\nu\nu'} = (K_a)^{\varepsilon\varepsilon'}_{\nu\nu'} + (\hat{S}_{0a})^{\varepsilon\varepsilon'}_{\nu\nu'} \quad (\text{A1})$$

where

$$(K_a)^{\varepsilon\varepsilon'}_{\nu\nu'} = P_{\nu}^{\varepsilon} K_a P_{\nu'}^{\varepsilon'}, \quad (\hat{S}_{0a})^{\varepsilon\varepsilon'}_{\nu\nu'} = P_{\nu}^{\varepsilon} \hat{S}_a P_{\nu'}^{\varepsilon'}. \quad (\text{A2})$$

Then action of the boost generator on ψ_ν^ε can be represented as:

$$J'_{0a}\psi_\nu^\varepsilon = (K_a)^{\varepsilon\varepsilon'}_{\nu\nu'}\psi_{\nu'}^{\varepsilon'} + (\hat{S}_{0a})^{\varepsilon\varepsilon'}_{\nu\nu'}\psi_{\nu'}^{\varepsilon'}. \quad (\text{A3})$$

where

$$\begin{aligned} (K_a)^{\varepsilon\varepsilon'}_{\nu\nu'} &= -p_0 \frac{\partial}{\partial p_a} \delta_{\varepsilon\varepsilon'} \delta_{\nu\nu'} + \frac{2ip_0}{p^2} S_{ab} p_b M_{\nu\nu'}^{\varepsilon\varepsilon'}, \\ (S_{0a})^{\varepsilon\varepsilon'}_{\nu\nu'} &= -\frac{\varepsilon p_a}{2p} \delta_{\varepsilon\varepsilon'} \delta_{\nu\nu'} + \frac{i\nu}{p} \gamma_5 S_{ab} p_b (2M_{\nu\nu'}^{\varepsilon\varepsilon'} - \delta_{\varepsilon\varepsilon'} \delta_{\nu\nu'}) \end{aligned} \quad (\text{A4})$$

and

$$M_{\nu\nu'}^{\varepsilon\varepsilon'} = \frac{1}{2} (\delta_{\varepsilon\varepsilon'} \delta_{\nu\nu'} + (\delta_{\varepsilon\nu} \delta_{\varepsilon'\nu'} - \delta_{\varepsilon\nu'} \delta_{\varepsilon'\nu}) \gamma_5). \quad (\text{A5})$$

The expressions (A4) for the entries of the boost generator can be calculated directly using definitions (9), (19) and (A2). In particular case $a = 3$ these expressions are reduced to the following form:

$$(K_3)^{\varepsilon\varepsilon'}_{\nu\nu'} = -p_0 \frac{\partial}{\partial p_3} \delta_{\varepsilon\varepsilon'} \delta_{\nu\nu'} + \frac{2ip_0}{p^2} (S_{31}p_1 + S_{32}p_2) M_{\nu\nu'}^{\varepsilon\varepsilon'}, \quad (\text{A6})$$

$$(S_{03})^{\varepsilon\varepsilon'}_{\nu\nu'} = -\frac{\varepsilon p_3}{2p} \delta_{\varepsilon\varepsilon'} \delta_{\nu\nu'} + \frac{i\nu}{p} \gamma_5 (S_{31}p_1 + S_{32}p_2) (2M_{\nu\nu'}^{\varepsilon\varepsilon'} - \delta_{\varepsilon\varepsilon'} \delta_{\nu\nu'}). \quad (\text{A7})$$

Since operators $(K_a)^{\varepsilon\varepsilon'}_{\nu\nu'}$ and $(\hat{S}_{0a})^{\varepsilon\varepsilon'}_{\nu\nu'}$ commute each other, the finite boost transformations produced by generator (A1) can be represented as a product of two transformations generated by these commuting parts of $(J_a)^{\varepsilon\varepsilon'}_{\nu\nu'}$:

$$p_\mu \rightarrow p'_\mu = \Lambda_{\mu\nu} p^\nu, \quad \psi_\nu^\varepsilon \rightarrow \psi'^\varepsilon_\nu \quad (\text{A8})$$

and

$$p'_\mu \rightarrow p''_\mu = p'_\mu, \quad \psi'^\varepsilon_\nu \rightarrow \psi''^\varepsilon_\nu \quad (\text{A9})$$

where (A8) and (A9) are the generic forms of transformation generated by $(K_a)^{\varepsilon\varepsilon'}_{\nu\nu'}$ and $(\hat{S}_{0a})^{\varepsilon\varepsilon'}_{\nu\nu'}$ correspondingly, Λ_ν^μ is a Lorentz transformation matrix.

To find the finite transformations of $\tilde{\psi}_\nu^\varepsilon$ generated by the infinitesimal operator (A6) it is sufficient to solve the following Lie equations:

$$\begin{aligned} \frac{\partial p'_3}{\partial \theta} &= -E', \\ p'_3|_{\theta=0} &= p_3; \end{aligned} \quad (\text{A10})$$

$$\begin{aligned} \frac{\partial \psi'^\lambda_\mu}{\partial \theta} &= \frac{i\tilde{p}p'_0}{p'^2} \Lambda_{\mu\nu}^{\lambda\varepsilon'} \psi'^{\varepsilon'}_{\nu'}, \\ \psi'^\lambda_\mu|_{\theta=0} &= \psi^\lambda_\mu \end{aligned} \quad (\text{A11})$$

where

$$\Lambda_{\mu\nu'}^{\lambda\varepsilon'} = \frac{2}{\tilde{p}}(S_{31}p_1 + S_{32}p_2)M_{\mu\nu'}^{\lambda\varepsilon'} \quad (\text{A12})$$

and $\tilde{p} = \sqrt{p_1^2 + p_2^2}$.

Since operator (A6) does not include derivations w.r.t. p_1 and p_2 , these variables are kept unchanged, i.e., $p'_1 = p_1$ and $p'_2 = p_2$. Thus $p'^2 = p_1'^2 + p_2'^2 + p_3'^2 = p_1^2 + p_2^2 + p_3'^2$.

Equations (A10) and (A11) can be integrated in closed form and give the following transformed functions:

$$\psi_\mu'^\lambda = \psi_\mu^\lambda + \left(\frac{p^2 + (2i(S_{31}p_1 + S_{32}p_2) + p_3)(p'_3 - p_3)}{pp'} - 1 \right) M_{\mu\nu'}^{\lambda\varepsilon'} \psi_{\nu'}^{\varepsilon'}. \quad (\text{A13})$$

To integrate (A11) we use the following observations:

- The θ -dependent multiplier in the r.h.s. of the first equation (A11) can be represented as:

$$\frac{p'_0}{p'^2} = \frac{\frac{\partial p'_3}{\partial \theta}}{\tilde{p}^2 + p_3'^2};$$

- Matrix Λ whose entries are given by equation (A12) does not depend on θ and satisfies the following conditions:

$$\Lambda^2 = M, \quad \Lambda^3 = \Lambda$$

where M is the matrix with entries (A5).

Thus we find explicitly the finite transformation (A8) generated by $(K_3)_{\mu\nu'}^{\lambda\varepsilon'}$. The next step is to find transformation (A9) generated by $(S_{03})_{\nu\mu}^{\varepsilon\lambda}$. The corresponding Lie equations reads:

$$\begin{aligned} \frac{\partial \psi_\nu''^\varepsilon}{\partial \theta} &= (S_{03})_{\nu\mu}^{\varepsilon\lambda} \psi_\mu'^\lambda, \\ \psi_\mu''^\lambda|_{\theta=0} &= \psi_\mu'^\lambda \end{aligned} \quad (\text{A14})$$

and have the following solution:

$$\psi_\nu''^\varepsilon = \left(\cosh \frac{\theta}{2} - \sinh \frac{\theta}{2} \frac{\varepsilon p_3}{p} \right) \delta_{\varepsilon\lambda} \delta_{\nu\mu} + i \sinh \frac{\theta}{2} \nu S_{3a} p_a \gamma_5 (\delta_{\varepsilon\nu} \delta_{\lambda\mu} - \delta_{\varepsilon\mu} \delta_{\lambda\nu}). \quad (\text{A15})$$

The completed boost transformation is a product of transformations (A13) and (A15). Substituting the expression (A13) for $\psi_\mu'^\lambda$ into (A15) we obtain equation (26).

References

- [1] Leanne D. Duffy, Karl van Bibber, *New J.Phys.* **11** (2009) 105008; arXiv:0904.3346.
- [2] A. G. Nikitin and Oksana Kuriksha, *Phys. Rev. D* **86** (2012) 025010; arXiv:1201.4935.

- [3] A. G. Nikitin and O. Kuriksha, *Commun. Nonlinear Sci. Numer. Simulat.* **17** (2012), 4585; arXiv:1002.0064.
- [4] D. V. Ahluwalia and D. Grumiller, *Phys. Rev. D* **72** (2005) 067701; arXiv:0410192.
- [5] D. V. Ahluwalia and D. Grumiller, *JCAP* **0507** (2005) 012; arXiv:0412.080.
- [6] C.G. Böhmmer, *Annalen der Physik* **16** (2007) 38; arXiv:0607.088.
- [7] Christian G. Böhmmer, David F. Mota, **Phys. Lett. B** **663** (2008) 168; arXiv:0710.2003.
- [8] A. Basak, J. R. Bhatt, S. Shankaranarayanan, K.V. P. Varma, *JCAP* **1304** (2013) 025; arXiv:1212.3445.
- [9] C. G. Böhmmer, *Annalen Phys.* **16** (2007) 325; ArXiv 0701.087.
- [10] L. Fabbri, *Phys. Lett. B* **704** (2011) 255; arXiv:1011.1637.
- [11] Y. X. Liu, X. N. Zhou, K. Yang and F. W. Chen, *Phys. Rev. D* **86** (2012): 064012; arXiv:1107.2506.
- [12] J. A Nieto, Higher Dimensional Elko Theory, arXiv:1307.1429.
- [13] P. Lounesto, in: *Clifford Algebras and Spinors*, 2nd ed., Chapters 11 and 12, p. 152-173, (Cambridge Univ. Press, Cambridge, 2002).
- [14] R da Rocha, WA Rodrigues Jr., *Mod. Phys. Lett. A* **21** (2006) 65; arXiv:0506075.
- [15] Cheng-Yang Lee, The Lagrangian for mass dimension one fermions. arXiv:1404.5307
- [16] D. V. Ahluwalia, C.-Y. Lee, and D. Schrittt, *Phys. Rev. D* **83** (2011) 065017, arXiv:0911.2947.
- [17] C. G. Boehmer, J. Burnett, D. F. Mota and D. J., *JHEP* **1007** (2010) 053; arXiv:1003.3858.
- [18] E. P. Wigner, *Ann. of Math.* **40** (1939) 149.
- [19] E. P. Wigner, *Unitary representations of the inhomogeneous Lorentz group including reflections*, in: *Group theoretical concepts and methods in elementary particle physics*, Ed. Gursey, F. (New York: Gordon and Breach 1964).
- [20] R. da Rocha and J. M. Hoff da Silva, *J. Math. Phys.* **48** (2007) 123517; arXiv:0711.1103.
- [21] D. V. Ahluwalia, S. P. Horvath, *JHEP* **11** (2010) 078; arXiv:1008.0436.
- [22] L. C. Biedenharn, M. Y. Han and H. van Dam, *Phys. Rev. D* **6** (1972) 500.
- [23] A. G. Nikitin, *J. Phys. A* **31** (1998) 13297.
- [24] J. Niederle and A. G. Nikitin, *J. Phys. A* **30** (1997) 999.
- [25] J. Beckers, N. Debergh, and A. G. Nikitin, *Int. J. Theor. Phys.* **36** (1997) 1991.

- [26] A. G. Nikitin, *Int. J. Mod. Phys. A* **14** (1999) 885.
- [27] J. Niederle and A. G. Nikitin, *J. Math. Phys.* **40** (1999) 1280.
- [28] V. X. Genest, J.-M. Lemay, L. Vinet, and A. Zhedanov, *J. Phys. A* **46** (2013) 505204, arXiv:1309.1701v1 2013.